The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
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Exploration 8: Altitudes and Orthocenters

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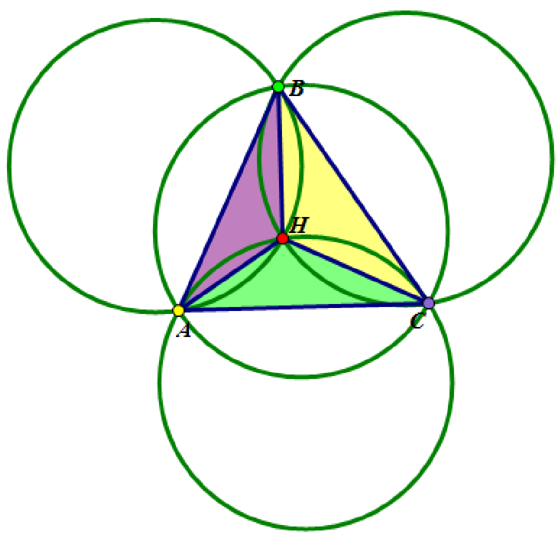
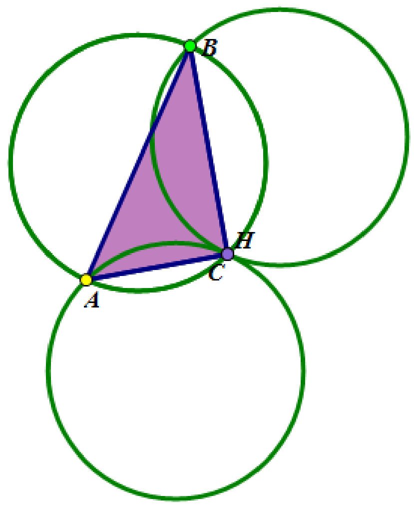
We will first construct a triangle ABC with orthocenter H. The orthocenter of triangle HBC is point A. The orthocenter of triangle HAB is point C. The orthocenter of triangle HAC is point B. In addition, we will construct the circumcircles of triangles ABC, HBC, HAB, and HAC.

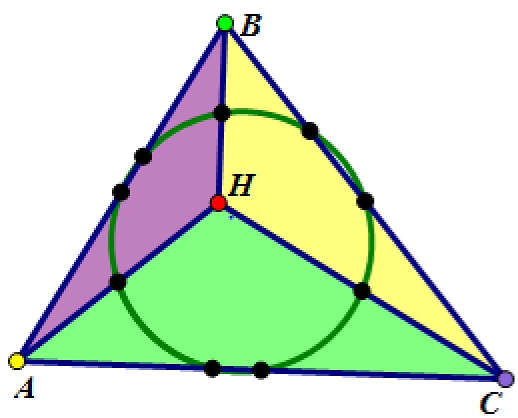


What would happen if any vertex of the triangle ABC was moved to where the orthocenter H is located?

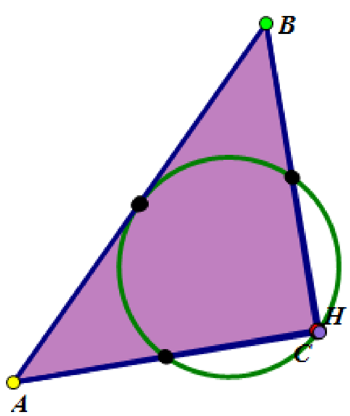


This shows us that the orthocenter of each sub-triangle is a vertex of the main triangle ABC.

We will now construct the nine point circles for triangles ABC, HBC, HAC, and HAB.



One interesting observation of this triangle is when any vertex of the triangle ABC is moved to where the orthocenter H is.

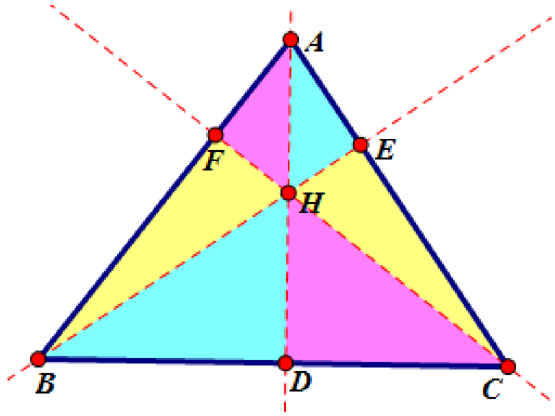


From this we see that the midpoint, feet of the altitude, and the midpoint of the line segment that connects each vertex to the orthocenter become equal.

Now let us explore the following:

Given acute triangle ABC. Construct the Orthocenter H. Let points D, E, and F be the feet of the perpendiculars from A, B, and C respectfully.

Prove: $\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}$ =1 and $\frac{AH}{AD}+\frac{BH}{BE}+\frac{CH}{CF}$ =2



We will make proofs based on the area of the triangles, similar to our proofs in exploration 4.

Prove: $\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}$ =1

Let us start with what we know. Area (ΔABC)=$\frac{1}{2}$*base*height=$\frac{1}{2}$*BC*AD. We can also re write this in two other ways depending on how we look at the triangle:

Area ΔABC=$\frac{1}{2}$*BC*AD=$\frac{1}{2}$*AC*BE=$\frac{1}{2}$*AB*CF.

We could also represent the area of ΔABC by taking the sum of the area of the three or six smaller triangles. Let's find the area of ΔABC by summing the area of the three smaller triangles ΔAHB, ΔAHC, ΔBHC.

Area ΔABC=$\frac{1}{2}$*AB*FH=$\frac{1}{2}$*AC*EH=$\frac{1}{2}$*BC*DH.

Using this information, let us create a ratio.

$\frac{Area\ \Delta\mathrm{ABC}}{Area\ \Delta\ \mathrm{ABC}}=\frac{\frac{1}{2}*\mathrm{AB}*\mathrm{FH}=\frac{1}{2}*\mathrm{AC}*\mathrm{EH}=\frac{1}{2}*\mathrm{BC}*\mathrm{DH}}{Area\ \Delta\mathrm{ABC}}$,

$1=\frac{\frac{1}{2}*\mathrm{AB}*\mathrm{FH}=\frac{1}{2}*\mathrm{AC}*\mathrm{EH}=\frac{1}{2}*\mathrm{BC}*\mathrm{DH}}{Area\ \Delta\mathrm{ABC}}$,

$1=\frac{\frac{1}{2}*\mathrm{AB}*\mathrm{FH}}{Area\ \Delta\mathrm{ABC}}+\frac{\frac{1}{2}*\mathrm{AC}*\mathrm{EH}}{Area\ \Delta\mathrm{ABC}}+\frac{\frac{1}{2}*\mathrm{BC}*\mathrm{DH}}{Area\ \Delta\mathrm{ABC}}$,

$1=\frac{\frac{1}{2}*\mathrm{AB}*\mathrm{FH}}{\frac{1}{2}*\mathrm{AB}*\mathrm{CF}}+\frac{\frac{1}{2}*\mathrm{AC}*\mathrm{EH}}{\frac{1}{2}*\mathrm{AC}*\mathrm{BE}}+\frac{\frac{1}{2}*\mathrm{BC}*\mathrm{DH}}{\frac{1}{2}*\mathrm{BC}*\mathrm{AD}}$,

$1 = \frac{HF}{CF} + \frac{EH}{BE} + \frac{HD}{AD}$,

$1 = \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF}$

Prove: $\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2$

We will begin by using the equation that we just proved.

$1 = \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF}$

We can represent the segment HD as AD-AH, and we can represent other segments in the same manner.

$1 = \frac{AD-AH}{AD} + \frac{BE-BH}{BE} + \frac{CF-CH}{CF}$,

$1 = \frac{AD}{AD} - \frac{AH}{AD} + \frac{BE}{BE} - \frac{BH}{BE} + \frac{CF}{CF} - \frac{CH}{CF}$,

$1 = 1 - \frac{AH}{AD} + 1 - \frac{BH}{BE} + 1 - \frac{CH}{CF}$,

$-2 = -1*(\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF})$,

$2 = \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF}$